Physics 604

Problem Set 1

Due Sept 16, 2010

1) a) Inside a good conductor the electric field is zero (electrons in the conductor, because they are free to move, move in a way to cancel any electric field impressed on the conductor, inside the volume of the conductor). Apply Gauss's law to any volume wholly inside the conductor.

$$\int_S \vec{E}\cdot\vec{n}da = 0 = \frac{q_{inside}}{\varepsilon_0} = 0$$

There can be no [net] charge inside the conductor. Electric field can come up to the surface of a conductor, where it must be terminated by a surface charge on the surface.

b) Suppose we have a closed conducting shell and a charge outside the shell. We would like to compute the electric field inside the shell because of the presence of the charge. Choose a closed surface completely inside the shell enclosing the interior of the shell. Because $\vec{E} = 0$ on the surface since it is inside the conductor, there is no net charge inside the surface, independent of where the exterior charge is placed, and no electric field induced inside of the surface by the exterior charge because the charge inside must remain zero. In other words, the interior of a conducting shell is shielded from external electrical disturbances. On the other hand, an unbalanced charge interior to the shell will generate a field outside the shell because

$$\int_S \vec{E}\cdot\vec{n}da = \frac{q_{inside}}{\varepsilon_0},$$

for any surface that totally encloses the shell.

c) A Gaussian pillbox argument shows $\vec{E}_{normal} = \sigma / \varepsilon_0$. If the pillbox extends inside the conductor to outside, and one considers a small area element ΔA

$$\int_{PB} \vec{E}\cdot\vec{n}da = \vec{E}_{normal}\Delta A - 0\Delta A = \frac{\sigma}{\varepsilon_0}\Delta A.$$

This specifies the normal component. Suppose one has a small Gaussian loop above the surface with displacement parallel to the surface. Then

$$0 = \int_L \vec{E}\cdot d\vec{l} \approx \vec{E}_t\Delta l - 0\Delta A \rightarrow \vec{E}_t = 0,$$

because the backward part of the loop integral, inside the conductor, must vanish. Because the loop can be oriented in any direction parallel to the surface, the tangential field (in all tangent directions) must vanish.

2) This problem is a straightforward application of Poisson's Equation. Note that for small *r*, the potential goes to

$$\Phi(r) \to \frac{q}{4\pi\varepsilon_0}\frac{1}{r} \equiv \Phi_{nucleus}(r)$$

The discussion in class shows

$$\nabla^2\Phi_{nucleus} = \frac{q}{\varepsilon_0}\delta(\vec{x})$$
$$\rho_{nucleus} = q\delta(\vec{x}),$$

representing the density of the singly-charged nucleus. There are no other singularities in the potential to worry about. Using the expression for the Laplacian in spherical coordinates on the back cover of the text,

$$-\frac{\rho(r)}{\varepsilon_0} = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left[\frac{qe^{-\alpha r}}{4\pi\varepsilon_0}\left(1+\frac{\alpha r}{2}\right)\right]$$
$$= \frac{q}{4\pi\varepsilon_0 r}\frac{\partial}{\partial r}\left[-\alpha e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)+e^{-\alpha r}\frac{\alpha}{2}\right]$$
$$= \frac{q}{4\pi\varepsilon_0 r}\left[\alpha^2 e^{-\alpha r}\left(1+\frac{\alpha r}{2}\right)-e^{-\alpha r}\frac{\alpha^2}{2}-e^{-\alpha r}\frac{\alpha^2}{2}\right]$$
$$= \frac{q}{4\pi\varepsilon_0}\frac{\alpha^3 e^{-\alpha r}}{2}.$$
$$\therefore \rho_{electron}(r) = -\frac{1}{4\pi}\frac{\alpha^3 e^{-\alpha r}}{2},$$

is the average electron charge distribution in the atom. Note that it integrates to the proper value:

$$q_{electron} = \int_V \rho_{electron} d^3x = -\frac{q}{4\pi}\iiint \frac{\alpha^3 e^{-\alpha r}}{2} r^2 dr\, d\cos\theta\, d\phi$$
$$= -\frac{q}{2}\int_0^\infty e^{-x}x^2 dx = -q\frac{\Gamma(3)}{2} = -q.$$

Because of the finite size of the nucleus, in fact the charge density of the proton is spread out, and the potential isn't $1/r$ at distances short compared to the proton size. This kind of thing is studied with great precision at Jefferson Lab.

3) a) Neglect the edge effects and assume that the charge is uniformly distributed on the plates, the upper plate with positive charge, the lower plate with negative charge. Assume the *z*-direction is aligned along d. By the usual Gaussian pillbox argument, the field inside is

$$E_z \Delta A = -\frac{\sigma}{\varepsilon_0}\Delta A \rightarrow E_z = -\frac{q}{\varepsilon_0 A}.$$

The potential difference is

$$\Delta\Phi = -\int \vec{E}\cdot d\vec{l} = -\int_0^d E_z dz = -E_z d\ .$$

The capacitance is

$$C = q/\Delta\Phi = \frac{\varepsilon_0 A}{d}.$$

b) Here assume that the negative charge is on the inside sphere, and uniformly distributed by symmetry. Gauss's Law gives

$$4\pi r^2 E_r = \frac{-q}{\varepsilon_0} \rightarrow E_r = \frac{-q}{4\pi\varepsilon_0}\frac{1}{r^2}.$$

The potential difference is

$$\Delta\Phi = -\int_a^b E_r dr = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right).$$

The capacitance is

$$C = \frac{4\pi\varepsilon_0 ab}{b-a}.$$

c) Assume that the negative charge is on the inside cylinder and uniformly distributed along the length of the cylinder. Gauss's Law gives

$$2\pi r E_r \Delta L = \frac{-(q/L)}{\varepsilon_0}\Delta L \rightarrow E_r = \frac{-(q/L)}{2\pi\varepsilon_0}\frac{1}{r}.$$

The potential difference is

$$\Delta\Phi = -\int_a^b E_r dr = \frac{q/L}{2\pi\varepsilon_0}(\ln b - \ln a).$$

The capacitance is

$$C = 2\pi\varepsilon_0 L / \ln(b/a).$$

d) One

4) a) Neglect the edge effects and the corrections due to the fact the charge is slightly displaced from uniform distribution on the conductor (corrections of order a/d and b/d), and assume that the fields are simply two line charge fields superposed.

$$\Phi(\vec{x}) \approx \frac{q/L}{2\pi\varepsilon_0}\ln\left|(\vec{x}-\vec{r}_a)/a\right| - \frac{q/L}{2\pi\varepsilon_0}\ln\left|(\vec{x}-\vec{r}_b)/b\right|$$

where $\vec{r}_{a,b}$ are the positions of the center of the line charges. To the first significant order in a/d and b/d,

$$\begin{aligned}\Delta\Phi(\vec{x}) &\approx \frac{q/L}{2\pi\varepsilon_0}\ln\left|d/a\right| - \frac{q/L}{2\pi\varepsilon_0}\ln\left|d/b\right| \\ &= \frac{q/L}{2\pi\varepsilon_0}\ln\left|\frac{d^2}{ab}\right| = \frac{q/L}{\pi\varepsilon_0}\ln\left|\frac{d}{\sqrt{ab}}\right|\end{aligned}$$

5) This problem is a straightforward application of Green's Second Identity. Following the same procedure as going from Eqn. 1.35 to Eqn. 1.36 yields (note that the ρ term is zero in Eqn. 1.36 because the scalar potential solves the Laplace Equation)

$$\begin{aligned}\Phi(\vec{x}) &= \frac{1}{4\pi}\int_S \left(\frac{1}{R}\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial}{\partial n'}\left(\frac{1}{R}\right)\right)da' \\ &= \frac{1}{4\pi a}\int_S \vec{\nabla}\Phi\cdot\vec{n}' da' + \frac{1}{4\pi}\int_S \frac{\Phi}{R^2}da' \\ &= \frac{1}{4\pi a}\int_V \nabla^2\Phi dx'dy'dz' + \frac{1}{4\pi a^2}\int_S \Phi da' \\ &= \frac{1}{4\pi a^2}\int_S \Phi(\vec{x}')da',\end{aligned}$$

where a is the radius of the sphere and $R = |\vec{x}-\vec{x}'|$. The final integral is clearly the average of the potential over the surface of the sphere. It does not matter what radius is chosen for the sphere in performing the average, but of course the values of the scalar potential on the surface *will* depend on the choice of radius.

6) The (upper bound) capacitance determined by the trial function is

$$
\begin{aligned}
C[\psi] &= \varepsilon_0 \int_V \left|\vec{\nabla}\psi\right|^2 d^3\vec{x} \\
&= \varepsilon_0 \frac{2\pi L}{(b-a)^2} \int_a^b r dr \\
&= \varepsilon_0 \frac{2\pi L}{(b-a)^2} \left[\frac{b^2}{2} - \frac{a^2}{2}\right] \\
&= \varepsilon_0 \pi L \frac{b+a}{b-a}
\end{aligned}
$$

where a is the inner radius of the cylinder, where b is the outer radius of the cylinder, and L is the length of the cylinder. Evaluating the exact and estimated capacitance numerically yields this table:

b/a	Exact $[C/2\pi\varepsilon_0 L] = (\ln b/a)^{-1}$	Trial Function $[C/2\pi\varepsilon_0 L] = (b/a+1)/(2b/a-2)$
1.5	2.46630	2.5
2	1.44270	1.5
3	0.91024	1.0

The "Exact" field is more like the linear trial function when $b \to a$. In the limit, clearly the two expressions agree by the expansion $\ln(1+x) \to x$ for small x. Notice the trial values are indeed higher than the exact values.